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The critical properties of the two-dimensional *xy* model

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Abstract. The critical properties of the *xy* model with nearest-neighbour interactions on a two-dimensional square lattice are studied by a renormalization group technique. The mean magnetization is zero for all temperatures, and the transition is from a state of finite to one of infinite susceptibility. The correlation length is found to diverge faster than any power of the deviation from the critical temperature. Analogues of the strong scaling laws are derived and the critical exponents, η and δ, are the same as for the two-dimensional Ising model.

1. Introduction

In a previous paper (Kosterlitz and Thouless 1973, to be referred to as I), we investigated the possibility of a phase transition in a number of two-dimensional systems with short-range interactions. In such systems the absence of conventional long-range order has been demonstrated rigorously by Mermin (1968), Mermin and Wagner (1966) and Hohenberg (1967). By studying the low-temperature behaviour on the basis of an iterated mean-field theory we showed that it is possible to define long-range order, called topological order, and that there is a phase transition characterized by a sudden change in the response of the system to an external perturbation.

In this paper we study in more detail the *xy* model of ferromagnetism with nearest-neighbour interactions, both above and below the critical temperature, and evaluate the critical exponents for the system. From work done on spin systems with lattices of higher dimension (Wilson 1971, Wilson and Fisher 1972, Wilson and Kogut 1972), we expect that the critical exponents will be independent of the details of the lattice structure and the range of the interaction, provided it is sufficiently short-ranged. The mechanism of the transition was explained in I. Besides the usual spin–wave excitations which are responsible for ensuring that the mean magnetization is zero at all temperatures, there are other equilibrium configurations which are not taken into account in the usual treatments. These are analogous to dislocations in a crystal or vortices in a superfluid and are characterized by the phase of the spins going through a multiple of 2π as we go round a contour enclosing the vortex. As discussed in I, the energy of an isolated vortex configuration of unit strength in such a spin system described by the hamiltonian

$$H_0 = -J \sum_{\langle ij \rangle} \mathbf{s}_i \cdot \mathbf{s}_j \tag{1.1}$$

is

$$E \approx 2\pi J \log R/\tau \tag{1.2}$$

where R is the radius of the system and τ the lattice spacing. For definiteness, we take the lattice to be square. Since there are $(R/\tau)^2$ possible positions for the vortex, its entropy is

$$S \approx 2k_B \log R/\tau + O(1) \tag{1.3}$$

where k_B is Boltzmann's constant. The free energy is

$$F \approx 2(\pi J - k_B T) \log R/\tau. \tag{1.4}$$

At sufficiently low temperatures, the energy dominates so that it is unfavourable for isolated vortices to occur, while at high temperatures the entropy term takes over and isolated vortices occur. The critical temperature is approximately

$$k_B T_c \approx \pi J. \tag{1.5}$$

In this crude approximation interactions between the vortices have been ignored and most of the rest of this paper is devoted to a detailed study of their effects on the critical behaviour.

Within the limitations of our approximations we confirm the hypothesis of Stanley (1968) that the susceptibility is infinite for all temperatures below the critical temperature and is finite above. We find that at $T \to T_c$, the correlation length ξ and the susceptibility χ diverge according to the asymptotic laws

$$\begin{aligned} \xi &\sim \exp(bt^{-1/2}) \qquad t > 0 \\ &= \infty \qquad\qquad\quad\; t < 0 \end{aligned} \tag{1.6}$$

where $t = (T - T_c)/T_c$ and $b \approx 1{\cdot}5$, and

$$\begin{aligned} \chi &\sim \xi^{2-\eta} \qquad t > 0 \\ &= \infty \qquad\quad t < 0 \end{aligned} \tag{1.7}$$

where the exponent $\eta = \frac{1}{4}$ has the same value as in the two-dimensional Ising model (Kaufman and Onsager 1949). This means that we cannot define the conventional exponents ν and γ since ξ diverges faster than any power of t. This is not inconsistent with some tentative results of Moore (private communication) who found that, on the basis of series expansions, γ did not settle to a constant value but continued to increase, indicating that the susceptibility diverges faster than a power of t. We further find that the singular part of the free energy behaves above T_c as

$$F \sim \xi^{-2} \qquad t > 0 \tag{1.8}$$

and that, in a weak external magnetic field h the magnetization m is asymptotically

$$m \sim h^{1/\delta} \qquad t = 0 \tag{1.9}$$

with $\delta = 15$. We thus see that the usual strong scaling laws are obeyed, provided they are expressed in terms of the correlation length, ie

$$\begin{aligned} \tilde{\gamma} &= 2 - \eta \\ \tilde{\alpha} &= -d \\ \delta &= (d + 2 - \eta)/(d - 2 + \eta) \end{aligned} \tag{1.10}$$

where $\tilde{\gamma}$ and $\tilde{\alpha}$ are defined by $\tilde{\gamma} = \gamma/\nu$, $\tilde{\alpha} = \alpha/\nu$ so that

$$\chi \sim \xi^{\tilde{\gamma}}$$

$$C_h \sim \xi^{\tilde{\alpha}}.$$

2. The model

The two-dimensional xy model is a system of spins confined to rotate in the plane of the lattice, which we take to be a simple square lattice with spacing τ. The hamiltonian of the system is

$$H_0 = -J \sum_{\langle ij \rangle} \boldsymbol{s}_i . \boldsymbol{s}_j = -J \sum_{\langle ij \rangle} \cos(\phi_i - \phi_j) \tag{2.1}$$

where $J > 0$ and the sum $\langle ij \rangle$ over the lattice sites is over nearest neighbours only. We have taken $|\boldsymbol{s}_i| = 1$, and ϕ_i is the angle the ith spin makes with some arbitrary axis. Only slowly varying contributions, that is those with adjacent angles nearly equal, will contribute significantly to the partition function. We may thus expand the hamiltonian about a local minimum up to terms quadratic in $\phi_i - \phi_j$.

$$H = H_0 - E_0 = \tfrac{1}{2}J \sum_{\langle ij \rangle} (\phi_i - \phi_j)^2 = J \int \mathrm{d}^2\boldsymbol{r}(\nabla\phi(\boldsymbol{r}))^2 \tag{2.2}$$

where we use a continuum notation for convenience. We expect that the critical properties of the system will be dominated by long-range effects where a continuum approximation will be sufficiently accurate.

As discussed in the Introduction, we must take into account the vortex configurations as well as the spin-wave excitations which are responsible for destroying the conventional long-range order. The vorticity q of a given region may be defined by

$$\oint \mathrm{d}\phi(\boldsymbol{r}) = 2\pi q \tag{2.3}$$

where the integral is taken round the boundary of the region. This suggests that we define

$$\phi(\boldsymbol{r}) = \psi(\boldsymbol{r}) + \bar{\phi}(\boldsymbol{r}) \tag{2.4}$$

where $\bar{\phi}(\boldsymbol{r})$ defines the angular distribution in the configuration of the local minimum of H and $\psi(\boldsymbol{r})$ defines the deviations from this. Clearly the absolute minimum of H is given by $\bar{\phi}(\boldsymbol{r}) = \text{const}$.

Since $\bar{\phi}(\boldsymbol{r})$ obeys Laplace's equation except at isolated points (the centres of the vortices), we can introduce the conjugate function $\bar{\phi}'(\boldsymbol{r})$ defined by

$$f(z) = \bar{\phi}(\boldsymbol{r}) + \mathrm{i}\bar{\phi}'(\boldsymbol{r}) \tag{2.5}$$

where $z = x + \mathrm{i}y$, $\boldsymbol{r} = (x, y)$. These obey the Cauchy–Riemann relations so that, using equation (2.3), we find

$$\nabla^2\bar{\phi}'(\boldsymbol{r}) = -2\pi\rho(\boldsymbol{r}) \tag{2.6}$$

where $\rho(\boldsymbol{r}) = \Sigma_i q_i \delta(\boldsymbol{r} - \boldsymbol{r}_i)$ is the distribution function of the vortices, and q_i is the strength of the ith vortex centred at $\boldsymbol{r}_i$.

The solution of equation (2.6) on a square lattice is

$$\bar{\phi}'(\boldsymbol{r}) = -2\pi \int \mathrm{d}^2\boldsymbol{r}'\rho(\boldsymbol{r}')\, g(\boldsymbol{r} - \boldsymbol{r}') + \mathrm{O}\left(\log(R/r_0) \int \rho(\boldsymbol{r})\, \mathrm{d}^2\boldsymbol{r}\right) \tag{2.7}$$

where $g(\boldsymbol{r})$ is the Green function on a two-dimensional square lattice defined so that $g(0) = 0$. An excellent approximation for $g(r)$ for all $r \geqslant \tau$ is (Spitzer 1964)

$$g(r) = \tfrac{1}{2}\pi \log\frac{r}{r_0} \tag{2.8}$$

where $\tau/r_0 = 2\sqrt{2}e^{\gamma}$. The second term in equation (2.7) demands that

$$\int \rho(\boldsymbol{r})\, d^2\boldsymbol{r} = 0 \qquad \text{or} \qquad \sum q_i = 0. \tag{2.9}$$

Thus, using equation (2.5), we see that $\bar{\phi}(\boldsymbol{r})$ is given by the imaginary part of

$$f(z) = -2\pi \sum_i q_i g(z - z_i) = -\sum_i q_i \log\left(\frac{z - z_i}{r_0}\right) \tag{2.10}$$

where the branch of $\log z$ is chosen so that

$$\log z = \log|z| + i \arg z. \tag{2.11}$$

Using the Cauchy–Riemann relations we find that

$$\int d^2\boldsymbol{r}(\nabla\phi(\boldsymbol{r}))^2 = \int d^2\boldsymbol{r}(\nabla\phi'(\boldsymbol{r}))^2 \tag{2.12}$$

so that for a configuration of $2n$ vortices in zero external field, from equations (2.6) to (2.8),

$$H_{2n}(\boldsymbol{r}_1 \ldots \boldsymbol{r}_{2n}) = J\int d^2\boldsymbol{r}(\nabla\psi(\boldsymbol{r}))^2 - 2\pi J \sum_{i \neq j}^{2n} q_i q_j \log\left|\frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{\tau}\right| + \mu \sum_i q_i^2 \tag{2.13}$$

where $2\mu q^2 = 4\pi J q^2 \log(\tau/r_0) \approx 2\pi^2 J q^2$ is the energy of a pair of vortices of opposite vorticity q separated by a single lattice spacing τ. The exact value of μ is unimportant as it affects only the numerical value of the critical temperature and has no effect on the form of the critical singularities. The only restriction on μ is that $\mu/k_B T_c$ is sufficiently large so that there is a low concentration of vortices near T_c in order to obtain a tractable theory. For the xy model, μ is determined by J, which in turn determines T_c so that $\mu/k_B T_c \approx \pi$ which is just large enough to obtain a selfconsistent theory. For simplicity we make the further assumption that only vortices of unit strength are present since these will be much more favourable than vortices with $|q| > 1$.

The partition function Z may now be written as

$$Z = \operatorname{Tr} \exp(-\beta H_{2n}) \tag{2.14}$$

where

$$\operatorname{Tr} \equiv \int \delta\psi(\boldsymbol{r}) \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \int_{D_{2n}} d^2\boldsymbol{r}_{2n} \ldots \int_{D_1} d^2\boldsymbol{r}_1 \tag{2.15}$$

and $\beta = 1/k_B T$. The functional integral over the angle $\psi(\boldsymbol{r})$ is carried out for $-\infty < \psi(\boldsymbol{r}) < +\infty$, although, strictly speaking $|\psi(\boldsymbol{r})| < \pi$. Since the hamiltonian is quadratic in $\psi(\boldsymbol{r})$, such a change should have a negligible effect. We must then further integrate over all possible positions of the vortices which are allowed to go over the whole plane subject only to the restriction $|\boldsymbol{r}_i - \boldsymbol{r}_j| \geqslant \tau$. Finally we sum over all possible numbers of vortices and include a statistical factor $(n!)^{-2}$ to allow for the fact that there are two sets of n vortices of equal but opposite vorticity.

3. The scaling equations

In this section we concentrate our attention on the contribution of the vortex configurations to the partition function. In the Appendix we show that, on scaling the lattice spacing τ to $\tau + d\tau$, we recover the partition function with unchanged functional form but with scaled interaction parameters given by equations (A.14) to (A.17). There is a fixed point solution to these equations at $K\tau^2 = 0$ and $\beta p^2 - 2 = 0$ which, in accordance with the hypothesis of Wilson (1971), we interpret as signalling the presence of a phase transition. Writing these equations in differential form and linearizing about the fixed point,

$$dx = -y^2 \frac{d\tau}{\tau} \tag{3.1}$$

$$dy^2 = -2xy^2 \frac{d\tau}{\tau} \tag{3.2}$$

$$dF = \frac{1}{8\pi} y^2 \frac{d\tau}{\tau^3} \tag{3.3}$$

where $x = \beta p^2 - 2 = 2(\pi\beta J - 1)$, $y = 4\pi K\tau^2$ and dF is the free energy due to the configurations where pairs of vortices are separated by $\tau < r < \tau + d\tau$, which have been averaged out.

These equations have an almost identical structure to those discussed by Anderson *et al* (1970, 1971), and are easily solved to obtain

$$x^2 - y^2 = x_0^2. \tag{3.4}$$

It can be shown that $x_0^2 = -Ct$ and a crude numerical calculation gives $C \approx 2{\cdot}1$ for the *xy* model. The trajectories of equation (3.4) are plotted in figure 1 for various values of t.

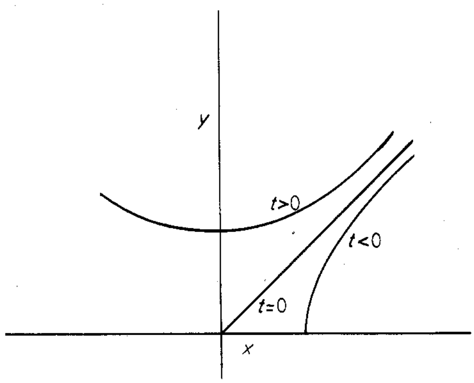

Figure 1. The trajectories of equation (3.4) for various values of t.

The critical temperature T_c is found by putting $x_0^2 = 0$ in equation (3.4), so that it is given by the solution of

$$\pi J/k_B T_c - 1 = 2\pi \exp(-\pi^2 J/k_B T_c). \tag{3.5}$$

This value of T_c is obtained by approximating $\epsilon(r) = \epsilon$ in the mean-field theory of I so that the effective energy of a pair of vortices is scaled by the same factor ϵ. We are unable to say which theory is more realistic, but the numerical differences are very small.

The solutions of the scaling equations for x and y as functions of τ have been obtained by Anderson *et al* (1970, 1971). In the region below T_c, x and y are positive for all τ, so that, using equations (3.1), (3.2) and (3.4) we find, for large τ

$$x(\tau) \approx x_0(1 + (\tau/\tau_i)^{-2x_0})(1 - (\tau/\tau_i)^{-2x_0})^{-1}. \tag{3.6}$$

At $T = T_c$, when $x_0 = 0$, this becomes

$$x(\tau) \approx (\log \tau/\tau_i)^{-1}. \tag{3.7}$$

We see that for sufficiently large τ, $x(\tau)$ has lost all knowledge of its initial value $x_i = x(\tau_i)$ provided only that $x_0 \ll x_i$. As $\tau \to \infty$, $x(\tau) \to x_0$. The same is true for $y(\tau)$ which tends to zero as $\tau \to \infty$. The behaviour of $x(\tau)$ and $y(\tau)$ is shown in figure 2. We interpret this as

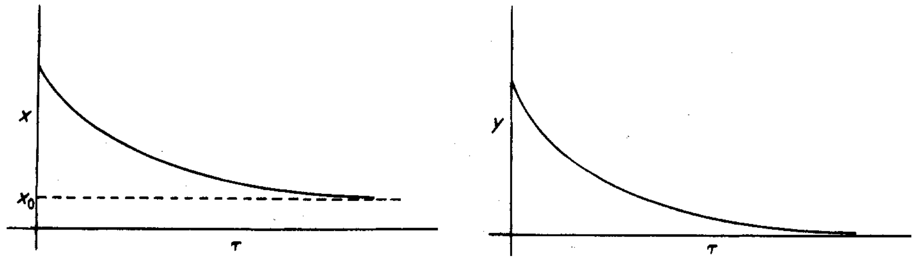

Figure 2. The scaling parameters x and y as functions of τ for $t < 0$.

meaning that, when the lattice spacing is scaled to infinity, all the vortices are scaled out of the problem since the chemical potential of a vortex becomes infinite. If we now identify the maximum value of τ before deviations from the fixed point become significant as the correlation length ξ, we have

$$\xi = \infty \qquad t < 0. \tag{3.8}$$

This is a rather surprising result but it has a simple physical explanation. The correlation between two spins is determined by two factors, the spin waves and the vortices. Below the critical temperature, the most favourable configuration of the vortices is for them to be bound together in close pairs of zero vorticity. The effect of a close pair on a spin far distant from the pair is very small since they almost cancel each other out. Even as $T \to T_c^-$ the mean separation of such pairs is finite (see I) so that as far as the average correlation of two widely separated spins is concerned, the vortices are still in small pairs. They have only a small effect and the correlation is determined by the spin waves which in turn implies an infinite correlation length.

Above the critical temperature. $t > 0$, the situation is not so clear cut. We make an analytic continuation $x_0 = iy_0$, y_0 real, so that $y_0^2 = Ct$, and now the solution of the scaling equations is

$$\frac{\tau}{\tau_i} = \exp\left\{\frac{1}{y_0}\left(\tan^{-1}\frac{y_0}{x} - \tan^{-1}\frac{y_0}{x_i}\right)\right\}. \tag{3.9}$$

We must choose the branch of $\tan^{-1}$ to be $0 < \tan^{-1} < \pi$ so that τ is an increasing function of x as x decreases from $x_i > 0$ through zero to negative values (see figure 3). For large values of τ equation (3.8) becomes

$$\frac{\tau}{\tau_i} \approx \exp\left(\frac{\pi}{y_0}\right) \exp\left\{-\left(\frac{1}{|x_{max}|} + \frac{1}{x_i}\right)\right\} \tag{3.10}$$

where $-|x_{max}| \ll -y_0$ is the value of x where deviations from the fixed point behaviour

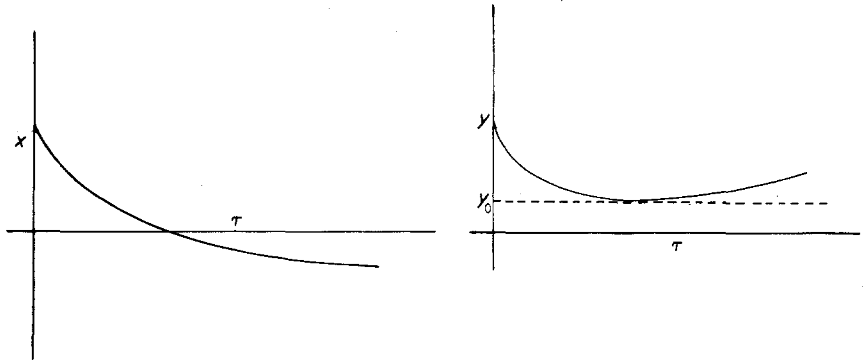

Figure 3. The scaling parameters x and y as functions of τ for $t > 0$.

become significant and the renormalization is stopped. Identifying the correlation length with the corresponding value of τ we obtain

$$\xi \underset{t \to 0^+}{\sim} \exp\left(\frac{\pi}{y_0}\right) \sim \exp(bt^{-1/2}) \tag{3.11}$$

where $b \approx 1{\cdot}5$. We thus have our first critical quantity ξ, which diverges as $t \to 0^+$ faster than any power of t and remains infinite for all $t \leqslant 0$. In this model, the exponent ν defined by $\xi \sim t^{-\nu}$ does not exist.

4. The two-point correlation function

In this section we study the two-point correlation function

$$\Gamma(r - r') = \langle s(r) . s(r') \rangle \qquad \text{for } |r - r'| \gg \tau \qquad \text{as } t \to 0^+.$$

The method used is very similar to that of the Appendix, so we only sketch the calculation. In terms of the angles $\phi(r)$, the two-spin correlation function is

$$\begin{aligned} \Gamma(r - r') &= \langle \cos(\phi(r) - \phi(r')) \rangle \\ &= \langle \cos(\bar{\phi}(r) - \bar{\phi}(r')) \cos(\psi(r) - \psi(r')) \rangle \\ &\equiv \Gamma_v(r - r')\,\Gamma_s(r - r') \end{aligned} \tag{4.1}$$

since the average $\langle \sin \psi \rangle$ vanishes. Fortunately, the two averages can now be carried out separately since H is a sum of two terms contaning $\bar{\varphi}(r)$ and $\psi(r)$ only.

Since H is quadratic in ψ, the average over ψ gives (Berezinskii 1970)

$$\Gamma_s(r) \underset{|r| > \tau}{\approx} \left|\frac{r}{\tau}\right|^{-\frac{1}{4\pi\beta J}}. \tag{4.2}$$

To evaluate $\Gamma_v(r)$ we write, explicitly displaying the cut-off,

$$\Gamma_v(\boldsymbol{r} - \boldsymbol{r}'; \tau) = \langle \exp \{2\pi \mathrm{i} \, \mathrm{Im} \sum_k q_k[g(z - z_k) - g(z' - z_k)]\}\rangle \tag{4.3}$$

where $q_k = \pm 1$, $\Sigma_k q_k = 0$ and we have used equation (2.10). This can be scaled by a slight generalization of the method described in the Appendix to obtain

$$\Gamma_v(\boldsymbol{r} - \boldsymbol{r}'; \tau) = \Gamma_v(\boldsymbol{r} - \boldsymbol{r}'; \tau + \mathrm{d}\tau) \exp\left\{-\frac{1}{8} y^2 \frac{\mathrm{d}\tau}{\tau} \log\left|\frac{\boldsymbol{r} - \boldsymbol{r}'}{\tau}\right|\right\} \tag{4.4}$$

where

$$\Gamma_v(\boldsymbol{r} - \boldsymbol{r}'; \tau + \mathrm{d}\tau) = \langle \exp \mathrm{i}(\bar{\phi}(\boldsymbol{r}) - \bar{\phi}(\boldsymbol{r}'))\rangle. \tag{4.5}$$

The average in equation (4.5) is taken with the hamiltonian appropriate to the scaled cut-off $\tau + \mathrm{d}\tau$. Equations (4.4) and (4.5) are sufficient to find the behaviour of $\Gamma_v(r)$.

Let us first consider the correlation function below the critical temperature. In this region, the correlation length is infinite and so we can continue the scaling procedure up to $\tau = r$, since at this stage the two spins under consideration are one lattice spacing apart. We then expect that

$$\Gamma_v(\boldsymbol{r} - \boldsymbol{r}'; \tau = |\boldsymbol{r} - \boldsymbol{r}'|) = \mathrm{O}(1) \tag{4.6}$$

so that the correlation function is given by

$$\Gamma_v(r) = \exp\left\{-\frac{1}{8}\int_{\tau_i}^{r} \frac{\mathrm{d}\tau}{\tau} y^2 \log\frac{r}{\tau}\right\} \tag{4.7}$$

where $r = |\boldsymbol{r} - \boldsymbol{r}'|$.

To evaluate the critical exponent η defined by $\Gamma(r) \underset{r\to\infty}{\sim} r^{-\eta}$ in two dimensions, we consider equation (4.7) for $t = 0$. Using the scaling equations (3.1), (3.2), (3.4) and (3.7) we have

$$\begin{aligned}\Gamma_v(r) &= \exp\left\{\tfrac{1}{8}\int_{x_i}^{x(r)} \mathrm{d}x \log r - \tfrac{1}{8}\int_{x_i}^{x(r)} \mathrm{d}x \log \tau\right\} \\ &\approx \exp\{\tfrac{1}{8}(x(r) - x_i)\log r - \tfrac{1}{8}\log x(r)\} \\ &\approx r^{-\frac{1}{8}x_i} (\log r)^{\frac{1}{8}} r^{-1/8 \log r}.\end{aligned} \tag{4.8}$$

The last factor in equation (4.8) is O(1), so that, ignoring the logarithmic term

$$\Gamma_v(r) \sim r^{-\frac{1}{8}x_i} \qquad t = 0. \tag{4.9}$$

Combining equations (4.2) and (4.9) and working to first order in x_i as we have been doing throughout, the total correlation function at $t = 0$ is

$$\Gamma(r) \sim r^{-\frac{1}{4}} \tag{4.10}$$

whence $\eta = \frac{1}{4}$. The value $\frac{1}{4}$ of the exponent η comes from a conspiracy between the spin-wave and vortex contributions. The spin-waves alone give $\eta = \frac{1}{4}k_B T_c/\pi J$. If we use the the simple free energy arguments of the Introduction we would find $\eta = \frac{1}{4}$ since, in this case, the noninteracting vortices do not contribute to η but serve only to determine $k_B T_c/\pi J = 1$. When we consider the interactions between the vortices, the critical temperature is reduced, but now the vortices contribute to η the exact amount required to bring η back to $\frac{1}{4}$. For $t < 0$, $\Gamma(r)$ falls off somewhat more slowly than $r^{-1/4}$, but its precise behaviour is not of interest. However, we have, from the fluctuation–dissipation theorem.

$$\chi \sim \int d^2\boldsymbol{r}\, \Gamma(r) = \infty \qquad t < 0. \tag{4.11}$$

These are two of the main results of this paper. In particular, the results $\eta = \frac{1}{4}$ is encouraging because the exact solution of the two-dimensional Ising model also gives $\eta = \frac{1}{4}$. This is not in complete agreement with our expectations as experience with the expansions of Wilson and Fisher (1972) indicate that the exponents depend on the number of degrees of freedom of the order parameter, so that one might expect that η for the xy model should differ slightly from $\frac{1}{4}$. However. we are working in a region where the methods of Wilson are inapplicable so when we are unable to say more about this question at this stage.

The situation for $t > 0$ is similar to that of the previous section in our discussion of the correlation length. We can explicitly follow the renormalization for $x_i > x(r) > -x > -1$, but when $x = O(1)$ our equations are no longer valid. However, in this region, as discussed by Anderson *et al*, we are in a very simple regime where $K\tau^2$ is large so that we have a very large number of vortices present and we are at very high temperatures. In effect, when we reach this stage we must calculate $\Gamma_v(r)$ for high temperatures.

By integrating equation (4.4) up to $\tau = \xi$ so that $-1 < x(\tau) < -y_0$, we find, for $r > \xi$,

$$\Gamma_v(r/\tau_i) = \Gamma'_v(r/\xi)\,(r/\xi)^{-\frac{1}{8}|x|}(r/\tau_i)^{-\frac{1}{8}x_i} \tag{4.12}$$

where Γ'_v is to be calculated with the hamiltonian appropriate for $\tau = \xi$. As argued above, this corresponds to calculating the correlation function in the high temperature régime where we expect it to fall off very rapidly with increasing values of r/ξ. We may thus write for $t > 0$

$$\Gamma(r) \sim r^{-\frac{1}{4}} f(r/\xi) \tag{4.13}$$

where $f(r/\xi)$ falls off rapidly for $r/\xi > 1$.

The susceptibility becomes

$$\chi \sim \int_{\xi}^{\infty} d^2r\, r^{-\frac{1}{4}} f(r/\xi) \sim \xi^{2-\frac{1}{4}}. \tag{4.14}$$

We have heuristic arguments which suggest that the contribution to χ from the region $r < \xi$ behaves like $\xi^{2-1/4-\alpha}$ where $\alpha \geqslant 0$ so that the susceptibility scales as equation (4.14).

5. The exponent δ

In a finite magnetic field h, the partition function becomes

$$Z = \operatorname{Tr} \exp\left\{-\beta H + \beta h \int d^2\boldsymbol{r} \cos\phi(\boldsymbol{r})\right\}. \tag{5.1}$$

The change in the free energy due to the magnetic field is

$$\beta\Delta F(h) \approx \left\langle \exp\left\{\beta h \int d^2r \cos\phi(\boldsymbol{r})\right\}\right\rangle. \tag{5.2}$$

Expanding equation (5.2), we obtain terms of the form

$$\int d^2\boldsymbol{r}_1 \ldots \int d^2\boldsymbol{r}_m (\beta h)^n \langle(\cos\phi(\boldsymbol{r}_1))^{p_1}\ldots(\cos\phi(\boldsymbol{r}_m))^{p_m}\rangle \tag{5.3}$$

where p_i are positive integers and $\Sigma_1^m p_i = n$. Writing $\cos\phi = \frac{1}{2}(\exp(\mathrm{i}\phi) + \exp(-\mathrm{i}\phi))$, we are led to consider averages of the form

$$\langle\exp\{\mathrm{i}\sum_k m_k\phi(\boldsymbol{r}_k)\}\rangle = \langle\exp\{\mathrm{i}\sum_k m_k\bar{\Phi}(\boldsymbol{r}_k)\}\rangle\langle\exp\{\mathrm{i}\sum_k m_k\psi(\boldsymbol{r}_k)\}\rangle. \tag{5.4}$$

Berezinskii (1970) has shown how to evaluate the average over the spin-wave excitations which can be done in the same way as for the two-point correlation function

$$\langle\exp\{\mathrm{i}\sum_k m_k\psi(\boldsymbol{r}_k)\}\rangle = \prod_{k\neq k'}\left|\frac{\boldsymbol{r}_k - \boldsymbol{r}_{k'}}{r_0}\right|^{-m_k m_{k'}/8\pi\beta J}\delta_{0,\Sigma m_k}. \tag{5.5}$$

By a simple generalization of the method used to scale the two-point correlation function we find

$$\langle\exp\{\mathrm{i}\sum_k m_k\bar{\Phi}(\boldsymbol{r}_k)\}\rangle_{(\tau)} = \langle\exp\{\mathrm{i}\sum_k m_k\bar{\Phi}(\boldsymbol{r}_k)\rangle_{(\tau+d\tau)}\exp\left\{\tfrac{1}{16}y^2\frac{d\tau}{\tau}\sum_{k\neq k'}m_k m_{k'}\log\left|\frac{\boldsymbol{r}_k - \boldsymbol{r}_{k'}}{\tau}\right|\right\} \tag{5.6}$$

where we explicitly display the cut-off by which the averages are taken. Integrating equation (5.6) at $t = 0$, and ignoring any logarithmic factors

$$\langle\exp\{\mathrm{i}\sum_k m_k\bar{\Phi}(\boldsymbol{r}_k)\}\rangle_{(\tau_i)} \sim \prod_{k\neq k'}\left|\frac{\boldsymbol{r}_k - \boldsymbol{r}_{k'}}{\tau_i}\right|^{\frac{1}{8}m_k m_{k'}x_i} \tag{5.7}$$

so that, to first order in x_i

$$\langle\exp\{\mathrm{i}\sum_k m_k\phi(\boldsymbol{r}_k)\}\rangle \sim \prod_{k\neq k'}\left|\frac{\boldsymbol{r}_k - \boldsymbol{r}_{k'}}{\tau_i}\right|^{\frac{1}{8}m_k m_{k'}}. \tag{5.8}$$

Consider now a term of the form (5.3) with all $p_i = 1$ and scale $\boldsymbol{r}_k = R\boldsymbol{r}'_k$, with R the radius of the system. Then the contribution to $\Delta F(h)$ from terms with all $p_i = 1$ is

$$\sum_{n=0}^{\infty} C_n(\beta h)^n \int d^2\boldsymbol{r}_1 \ldots \int d^2\boldsymbol{r}_n \prod_{k\neq k'}\left|\frac{\boldsymbol{r}_k - \boldsymbol{r}_{k'}}{\tau_i}\right|^{\frac{1}{8}m_k m_{k'}}$$

$$= \sum_{n=0}^{\infty} C_n(\beta h)^n R^{n(2-\frac{1}{8})}\int_{|\boldsymbol{r}_1|<1} d^2\boldsymbol{r}_1 \ldots \int_{|\boldsymbol{r}_n|<1} d^2\boldsymbol{r}_n \left|\frac{\boldsymbol{r}_k - \boldsymbol{r}_{k'}}{\tau_i}\right|^{\frac{1}{8}m_k m_{k'}}. \tag{5.9}$$

We now demand that, as $R \to \infty$, the free energy per unit area has a finite limit

$$\frac{\Delta F(h)}{R^2} = \Delta f(h) \tag{5.10}$$

and look for solutions of the form

$$\Delta F(h) = a_0 u^{b_0} \tag{5.11}$$

where $u = \beta h R^{(2-\frac{1}{8})}$. Using equations (5.10) and (5.11), we find that

$$\Delta f(h) \sim h^{16/15}. \tag{5.12}$$

By similar arguments it can be shown that the terms (5.3) with any $p_i > 1$ give contributions to $\Delta f(h)$ which vanish with a higher power of h as $h \to 0$. Provided the series converges

$$\Delta f(h) \underset{h\to 0}{\sim} h^{16/15}. \tag{5.13}$$

Thus, the mean magnetization $m \sim h^{1/\delta}$ with $\delta = 15$.

6. Free energy and thermal properties

For $t < 0$, the free energy is calculated in exactly the same way as in Anderson *et al* (1970, 1971) by integrating equation (3.3).

$$F_{t<0} \sim \exp\left\{\frac{-\text{const}}{t^{1/2}}\right\} + \text{analytic part.}$$

Similarly, for $t > 0$, remembering that we must cut the integrals off at $x \sim -1$

$$F_{t>0} \sim \exp(-2\pi/y_0) + \text{analytical part.}$$

Thus, the specific heat is regular at $t = 0$ since all derivatives of the singular part of F vanish at $t = 0$. As expected, as $t \to 0^+$

$$F_{\text{sing}} \sim \xi^{-2}.$$

Appendix

We derive the scaling equations for the system in zero magnetic field and discuss the relation to the two-dimensional classical plasma consisting of n positively and n negatively charged particles interacting with the usual logarithmic Coulomb potential. The contribution of a vortex configuration to the hamiltonian is

$$H_{2n}(\boldsymbol{r}_1 \ldots \boldsymbol{r}_{2n}) = -\sum_{i\neq j} p_i p_j \log\left|\frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{\tau}\right| + 2n\mu \tag{A.1}$$

where $p_i = \pm(2\pi J)^{1/2}$ and $\Sigma p_i = 0$. This has an identical structure to the hamiltonian for the classical plasma with charges $p_i = \pm p$ except for the chemical potential μ. However, if we form the grand partition function for such a system

$$Z = \sum_n \frac{1}{(n!)^2} K^{2n} \int d^2 \boldsymbol{r}_{2n} \ldots \int d^2 \boldsymbol{r}_1 \exp\{-\beta H_{2n}\} \tag{A.2}$$

where $K\tau^2 = \exp(-\beta\mu)$ and with the restriction that the system is dilute, so that $\mu/k_B T$ is large, we have precisely the same problem as before. In this case τ is the diameter of the particles. One may worry about the fact that the integrations over the $\boldsymbol{r}_i$ restrict the range to $|(\boldsymbol{r}_i - \boldsymbol{r}_j)/\tau| > 1$, but one can always redefine the size of the particles by suitably redefining μ so that this restriction still holds. Provided the system is sufficiently dilute, the condition that $\mu/k_B T$ is large will still hold. Thus, if we can solve the problem of the two-dimensional classical plasma we solve the xy model at the same time.

For generality, we include an electric field $\boldsymbol{E}$, which has no easily realizable physical analogue in the xy model, so that the partition function becomes

$$Z = \sum_n \frac{1}{(n!)^2} K^{2n} \int_{D_{2n}} d^2 \boldsymbol{r}_{2n} \ldots \int_{D_1} d^2 \boldsymbol{r}_1 \exp\left\{\beta \sum_{i \neq j} p_i p_j \log \left|\frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{\tau}\right| + \beta \sum p_i \, \boldsymbol{E} . \boldsymbol{r}_i\right\} \quad \text{(A.3)}$$

where we keep the order of the integrations fixed and D_k is the whole plane except for the circles $|\boldsymbol{r}_k - \boldsymbol{r}_j| < \tau$, $j = k + 1, k + 2, \ldots, 2n$. In I we investigated this by an iterated mean-field approximation which yielded a solution to the problem for $T < T_c$, but did not give much information for $T > T_c$.

Anderson *et al* (1970, 1971) have studied a similar partitition function for the one-dimensional case corresponding to the Kondo problem and the Ising model with interactions falling off at long distances as r^{-2}. Their methods are directly applicable to our problem. The idea is to scale the lattice spacing or minimum particle separation from τ to $\tau + d\tau$ with $d\tau \ll \tau$ and find the partition function for this new problem which should be of the same functional form as the initial one. To $O(d\tau)$ we can rearrange the integrals of equation (A.3) as

$$\int_{D_{2n}} d^2 \boldsymbol{r}_{2n} \ldots \int_{D_1} d^2 \boldsymbol{r}_1 = \int_{D'_{2n}} d^2 \boldsymbol{r}_{2n} \ldots \int_{D'_1} d^2 \boldsymbol{r}_1 + \tfrac{1}{2} \sum_{i \neq j} \int_{D'_{2n}} d^2 \boldsymbol{r}_{2n} \ldots \int_{D'_{j+1}} d^2 \boldsymbol{r}_{j+1}$$
$$\times \int_{D'_{j-1}} d^2 \boldsymbol{r}_{j-1} \ldots \int_{D'_{i+1}} d^2 \boldsymbol{r}_{i+1} \int_{D'_{i-1}} d^2 \boldsymbol{r}_{i-1} \ldots \int_{D'_1} d^2 \boldsymbol{r}_1 \int_{\bar{D}_j} d^2 \boldsymbol{r}_j \int_{\delta_i(j)} d^2 \boldsymbol{r}_i \quad \text{(A.4)}$$

where D'_k is defined as for D_k except that τ is replaced by $\tau + d\tau$, $\delta_i(j)$ is the annulus $\tau < |\boldsymbol{r}_i - \boldsymbol{r}_j| < \tau + d\tau$, and $\bar{D}_j$ is the whole plane except for the circles of radius τ about the other $2n - 2$ points, $\boldsymbol{r}_k$, $k \neq i, j$. The integral over the domain $\delta_i(j)$ selects particle i and pairs it with particle j. The integral over $\bar{D}_j$ then takes this pair over the whole allowed region of the plane. The sum $\frac{1}{2}\Sigma_{i \neq j}$ selects each possible pair once. The main contribution to Z will come from pairing particles with opposite charge so that may restrict the sum over i and j in equation (A.4) to be overparticles with $p_i = -p_j$. These three operations have the effect of averaging out the contribution to Z of all close pairs or particles, thereby producing a new problem where the minimum spacing is $\tau + d\tau$.

To proceed, we must carry out the integrations over $\delta_i(j)$ and $\bar{D}_j$. We select all the terms in $H(\boldsymbol{r}_1 \ldots \boldsymbol{r}_{2n})$ containing $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$ and consider

$$\int_{\delta_i(j)} d^2 \boldsymbol{r}_i \exp\left\{2\beta\left(\sum_k p_i p_k \log\left|\frac{\boldsymbol{r}_i - \boldsymbol{r}_k}{\tau}\right| + \sum_k p_j p_k \log\left|\frac{\boldsymbol{r}_j - \boldsymbol{r}_k}{\tau}\right|\right)\right.$$
$$\left. + \beta \boldsymbol{E} . (p_i \boldsymbol{r}_i + p_j \boldsymbol{r}_j)\right\}. \quad \text{(A.5)}$$

Since $p_i = -p_j$ and $\boldsymbol{r}_i - \boldsymbol{r}_j = \boldsymbol{\tau}$ we obtain

$$\tau \, d\tau \int_0^{2\pi} d\theta \prod_k \left(1 + \frac{2\boldsymbol{\tau} . (\boldsymbol{r}_j - \boldsymbol{r}_k)}{|\boldsymbol{r}_j - \boldsymbol{r}_k|^2} + \frac{\tau^2}{|\boldsymbol{r}_j - \boldsymbol{r}_k|^2}\right)^{\beta p_i p_k} \exp\{\beta p_i \boldsymbol{E} . \boldsymbol{\tau}\} \quad \text{(A.6)}$$

where θ is the angle $\boldsymbol{\tau}$ makes with some axis. Using the fact that, provided $K\tau^2$ is sufficiently small, we can expand out the product in equation (A.6) to $O(\tau^2/|\boldsymbol{r}_j - \boldsymbol{r}_k|^2)$ since the probability of having three particles (remember that there is a neutral pair at $\boldsymbol{r}_j$) close together (ie $|\boldsymbol{r}_j - \boldsymbol{r}_k| = O(\tau)$) is very low, so that the main contribution to Z will come from the regions where $|\boldsymbol{r}_j - \boldsymbol{r}_k| \gg \tau$. Carrying out the expansion to this order and to terms linear in $\boldsymbol{E}$ and doing the integrals over θ we obtain

$$2\pi\tau\,\mathrm{d}\tau\left\{1 + \beta^2 p^4 \sum_k \frac{\tau^2}{|\boldsymbol{r}_j - \boldsymbol{r}_k|^2} + \beta^2 p^2 \sum_{k \neq l} p_k p_l \frac{\tau^2(\boldsymbol{r}_j - \boldsymbol{r}_k).(\boldsymbol{r}_j - \boldsymbol{r}_l)}{|\boldsymbol{r}_j - \boldsymbol{r}_k|^2 |\boldsymbol{r}_j - \boldsymbol{r}_l|^2} + \beta^2 p^2 \sum_k p_k \frac{\tau^2 \boldsymbol{E}.(\boldsymbol{r}_j - \boldsymbol{r}_k)}{|\boldsymbol{r}_j - \boldsymbol{r}_k|^2}\right\}. \tag{A.7}$$

The integration of equation (A.7) over $\bar{D}_j$ is tedious but elementary and our final result is

$$2\pi\tau\,\mathrm{d}\tau\left[A - 2\pi\tau^2\beta^2 p^2 \sum_{k\neq l} p_k p_l \log\left|\frac{\boldsymbol{r}_k - \boldsymbol{r}_l}{\tau}\right|\right] \tag{A.8}$$

where A is the area of the system.

Using equations (A.4) and (A.8) and rearranging the summation over n we may write the partition function as

$$Z = \sum_n \frac{1}{(n!)^2} K^{2n} \int_{D'_{2n}} \mathrm{d}^2 r_{2n} \ldots \int_{D'_1} \mathrm{d}^2 r_1 \left[1 + \frac{K^2}{(n+1)^2} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} 2\pi\tau\,\mathrm{d}\tau \times \left(A - 2\pi\tau^2\beta^2 p^2 \sum_{k\neq l} p_k p_l \log\left|\frac{\boldsymbol{r}_k - \boldsymbol{r}_l}{\tau}\right|\right)\right] \exp\left\{\beta \sum_{k\neq l} p_k p_l \log\left|\frac{\boldsymbol{r}_k - \boldsymbol{r}_l}{\tau}\right| + \beta \sum_k p_k\, \boldsymbol{E}.\boldsymbol{r}_k\right\}. \tag{A.9}$$

We notice that on integrating equation (A.5) over the domains $\delta_i(j)$ and $\bar{D}_j$ all dependence on the labels i and j has dropped out so that the summation over i and j in equation (A.9) simply gives a factor $(n+1)^2$. Since we are working to $\mathrm{O}(\mathrm{d}\tau)$ we may take the exponential of the factor in square brackets in equation (A.9) so that the partition function becomes

$$Z = \exp\{2\pi K^2 \tau\,\mathrm{d}\tau A\} \sum_n \frac{1}{(n!)^2} K^{2n} \int_{D'_{2n}} \mathrm{d}^2 r_{2n} \ldots \int_{D'_1} \mathrm{d}^2 r_1 \exp\left\{\beta\left(1 - (2\pi)^2 \beta p^2 \times (K\tau^2)^2 \frac{\mathrm{d}\tau}{\tau}\right) \sum_{i\neq j} p_i p_j \log\left|\frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{\tau}\right| + \beta \sum_i p_i\, \boldsymbol{E}.\boldsymbol{r}_i\right\}. \tag{A.10}$$

The last step is to change $\tau \to \tau + \mathrm{d}\tau$ in the integrand of equation (A.10) since this will bring the partition function into exactly the same functional form as the initial one except that the cut-off, or minimum particle separation, has been increased to $\tau + \mathrm{d}\tau$. This can easily be seen to change the activity $K\tau^2$ since the relevant part of equation (A.10) transforms as

$$K^{2n} \exp\{-\beta \sum_{i\neq j} p_i p_j \log \tau\} = \left[K\left(1 - \beta p^2 \frac{\mathrm{d}\tau}{\tau}\right)\right]^{2n} \exp\left\{-\beta \sum_{i\neq j} p_i p_j \log(\tau + \mathrm{d}\tau)\right\}. \tag{A.11}$$

We now have

$$Z(\beta p^2, K\tau^2) = Z_0 Z(\widetilde{\beta p}_2, \widetilde{K\tau^2}) \tag{A.12}$$

where $Z(\widetilde{\beta p}^2, \widetilde{K\tau}^2)$ is the partition function for the new problem with cut-off $\tau + d\tau$ and

$$Z_0 = \exp\{2\pi K^2 \tau \, d\tau A\} \tag{A.13}$$

is the partition function of the pairs with separations between τ and $\tau + d\tau$ which have been averaged out. From equations (A.10) and (A.11), we see that the interaction parameters βp^2, $K\tau^2$ and Ep have been scaled according to

$$\beta p^2 \to \widetilde{\beta p}^2 = \beta p^2 \left(1 - (2\pi)^2 (\beta p^2)(K\tau^2)^2 \frac{d\tau}{\tau}\right) \tag{A.14}$$

$$K\tau^2 \to \widetilde{K\tau}^2 = K\tau^2 \left(1 - (\beta p^2 - 2)\frac{d\tau}{\tau}\right) \tag{A.15}$$

$$Ep \to \widetilde{Ep} = Ep. \tag{A.16}$$

The free energy per unit area of the particles which have been averaged out is

$$dF = 2\pi K^2 \tau \, d\tau. \tag{A.17}$$

Equations (A.14), (A.15) and (A.17) are those used in § 3, while equation (A.16) is relevant only to the plasma. Exactly as for the xy model, as we scale $\tau \to \infty$ for $T < T_c$, the total free energy is given by integrating equation (A.17) so that the specific heat is regular as $T \to T_c^-$. Similar arguments above T_c show that the specific heat is also regular as $T \to T_c^+$. The critical temperature T_c is given by the solution of the equation

$$\frac{p^2}{k_B T_c} - 2 = 2\pi \exp\left(-\frac{\mu}{k_B T_c}\right) \tag{A.18}$$

obtained by integrating equations (A.14) and (A.15). The behaviour of the dielectric constant ϵ is found from equation (A.16), which on integrating gives

$$E(\tau) = E(\tau_i) \exp\{\tfrac{1}{4}(x_i - x(\tau))\} \tag{A.19}$$

where $x = \beta p^2 - 2$. When $\tau \to \infty$, all the particles are scaled out of the problem, so that the dielectric constant $\epsilon(\infty) = 1$ and the actual dielectric constant $\epsilon = \epsilon(\tau_i)$ is given by

$$E(\infty) = \epsilon(\tau_i) E(\tau_i) \tag{A.20}$$

so that

$$\epsilon = \exp \tfrac{1}{4}(x_i - x_0) \approx 1 + \tfrac{1}{4}(x_i - x_0) \tag{A.21}$$

for $x \ll 1$. Remembering that $x_0 \propto (T_c - T)^{1/2}$, the dielectric constant approaches its critical value as

$$\epsilon(T_c) - \epsilon(T) \sim (T_c - T)^{1/2} \tag{A.22}$$

which is to be compared with the singularity obtained in I by the mean-field approximation. Within the context of a mean-field theory, equation (A.21) is obtained by assuming that the effective energy of a pair of oppositely charged particles is

$$E_{\text{eff}} = \frac{2p^2}{\epsilon} \log r/\tau$$

while in I we assumed that ϵ was dependent upon r, the distance between the particles. We cannot say at this stage which theory is more accurate.

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